Feasibility of loophole-free nonlocality tests with a single photon

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Recently much interest has been directed towards designing setups that achieve realistic loss thresholds for decisive tests of local realism, in particular in the optical regime. We analyse the feasibility of such Bell tests based on a W-state shared between multiple parties, which can be realised for example by a single photon shared between spatial modes. We develop a general error model to obtain thresholds on the efficiencies required to violate local realism, and also consider two concrete optical measurement schemes.

I. INTRODUCTION

Local realism – the assumption that physical quantities have well established values previous to any measurement and that signals cannot travel faster than the speed of light – entails limits on the correlations which space-like separated, independent observers may obtain. Such restrictions, usually expressed as Bell inequalities [1], may be surpassed within quantum theory when the observers share certain entangled quantum states. The correlations then cannot be explained by any local hidden variable (LHV) model, and are hence labeled nonlocal correlations.

The experimental evidence for the existence of nonlocal correlations is compelling yet not completely conclusive. Existing tests all suffer from one or more loopholes, which open up LHV explanations of the data unless further assumptions are introduced. In some cases, the measurements are not sufficiently fast or far apart to be space-like separated events [2]. In other cases, low detection efficiency is the obstacle. For example, in Bell tests based on photon pairs [3], low photodetection efficiencies imply that in many experimental runs, at least one photon is lost. Analysing only the coincidence counts, nonlocal correlations may be observed even when the total dataset can be explained by a LHV model. An additional fair-sampling assumption is therefore required to reject local realism in experiments with low detection efficiency. While this may seem natural, albeit slightly dissatisfactory, in experiments probing the nature of quantum mechanics, it is incompatible with device independent applications for which the violation of a Bell inequality is a necessary condition to assure e.g. security of certain tasks [4–6]. For example, a malicious eavesdropper could take advantage of detection inefficiency to break a device-independent key distribution protocol without being detected.

The robustness of Bell test violations to loss and inefficiencies depends on the particular test and setup. To achieve efficiency thresholds compatible with those of laboratory detectors, various approaches have been considered, e.g. changing the number of settings, outcomes and parties [7–12], as well as the detection schemes and types of states [12–17]. Interestingly, as first noted by Eberhard [18], the state providing better robustness against losses, is not necessarily the most entangled. For the CHSH inequality, corresponding to 2 parties, 2 settings, and 2 outcomes, a critical efficiency of 67% is achieved by an almost separable state. However, such a state is very susceptible to noise, and hence not terribly practical in experiment.

Recently, asymmetric setups have attracted attention. Asymmetry in the efficiency of different measurement settings is motivated by the fact that the measurements which can be efficiently performed do not necessarily coincide with those yielding a high Bell test violation. For example, homodyne detection of light can be extremely efficient, but it seems very difficult to obtain good violations based solely on homodyning [19]. Combining homodyning and single-photon detection, in Ref. [15] the authors obtain a threshold of 71% for single-photon detection. This is higher than for the Eberhard scheme, however the required state is not close to separable and seems feasible to prepare. Asymmetry between parties is motivated by the natural occurence of entanglement between different physical systems, such as atoms and photons, for which the available detection schemes have widely different efficiencies. Atomic states can be measured with almost unit efficiency. Nonlocality tests for atom-photon systems have been investigated in Refs. [12, 13], where a critical single-photon detection efficiency of 45% was obtained, and very recently in Ref. [17], where 39% was obtained for a scheme combining atoms, single-photon and homodyne detection.

In this paper, we consider general asymmetric Bell tests, based on multipartite W-states [20]. The physical implementation we have in mind is a single photon shared between multiple parties, and possibly entangled with an additional atomic system. However our analysis is applicable also to a wider qubit setting. The motivation for focusing on nonlocality tests based on W-states is threefold: the simplicity of preparing single-photon W-states, the robustness of W-state entanglement against losses, and the existence of Bell inequality violations which rapidly approaches the algebraic maximum for an increasing number N of parties.

In the following sections we first discuss optical W-states and examine concrete measurements schemes in Sec. II. We introduce a POVM model capturing a broad range of detection imperfections and loss in Sec. III, and

in Secs. IV and V we explore the robustness of nonlocality for the W-state by means of specific Bell inequalities and linear programming respectively. We demonstrate, using the Bell inequality proposed in Ref. [21], that the local content of the W-state tends exponentially fast to zero for increasing N. This, and the fact that the entanglement in the state has a very robust (size-independent) resistance to losses [22], would seem to suggest that there could be an advantage to increasing N in tests of nonlocality. However, we show that this is not the case for several concrete examples where the robustness actually decreases with N, while in other cases which we have examined, only slight improvements with N are found. At the same time the threshold scaling, for the cases with no improvements, is not severe. This is positive, since multipartite nonlocality is important in its own right and useful in several information processing tasks [23].

II. STATE AND PHYSICAL MEASUREMENTS

The W-state for N parties, each a 2-dimensional quantum system, can be taken to be

$$|W_N\rangle = \frac{1}{\sqrt{N}}\left(|10\ldots0\rangle + \cdots + |0\ldots01\rangle\right), \qquad (1)$$

where $|0\rangle$, $|1\rangle$ denote the eigenstates with eigenvalues 1, -1 respectively of the Pauli operator $\hat{\sigma}_z$. In a purely optical implementation, (1) can be created by coherently distributing a single photon among N parties. Heralded single photons can be generated e.g. via spontaneous parametric down conversion, and distributed by means of linear optics. Measurements of $\hat{\sigma}_z$ then correspond simply to single-photon detection (SPD). However, measurements in the x-y plane of the Bloch sphere require transformations such as $|0\rangle + |1\rangle \to |0\rangle$ that do not preserve energy, and hence cannot be implemented perfectly with passive, linear optics. Nevertheless, approximate implementations are possible as discussed below.

Another case we will consider is atom-photon entanglement. For single trapped atoms or ions, very high detection efficiency and good control over the measurement bases can be achieved [2]. At the same time, through spontaneous emission, the atom can be entangled with an optical field, e.g. in a state of the form $\cos(\theta)|e,0\rangle + \sin(\theta)|g,1\rangle$, with the first mode referring to the atomic state and the second to the number of photons in the field. Distributing the emitted photon over multiple modes, one arrives at a W-state with one atomic party and $N-1$ photonic parties, that is

$$|W_{asym}\rangle = \cos(\theta)|e\rangle|vac\rangle_{N-1} + \sin(\theta)|g\rangle|W\rangle_{N-1}, \qquad (2)$$

where $|vac\rangle$ denotes a state with all modes in $|0\rangle$. Since the coupling η_c between the atom and the emitted photon is not perfect, the actual state will be an incoherent sum of $\cos(\theta)|e\rangle|vac\rangle_{N-1} + \sqrt{\eta_c}\sin(\theta)|g\rangle|W\rangle_{N-1}$ and $|g\rangle|vac\rangle_{N-1}$ with respective probabilities given by $\cos^2(\theta) + \eta_c\sin^2(\theta)$ and $(1-\eta_c)\sin^2(\theta)$.

A. Pauli measurement via homodyning

An approximate implementation of $\hat{\sigma}_x$ can be achieved using the fact that in the 0-1 photon Fock space, a homodyne measurement with phase ϕ and sign-binning approximates a measurement of $\cos(\phi)\hat{\sigma}_x + \sin(\phi)\hat{\sigma}_y$, i.e. in the equatorial plane of the Bloch sphere [16]. For example, within the 0-1 photon Fock space, the elements of the projection operator approximating $\hat{\sigma}_x$ are

$$\begin{aligned}\pi^x_{nm} &= \langle n|\left(\int_0^\infty dx|x\rangle\langle x| - \int_{-\infty}^0 dx|x\rangle\langle x|\right)|m\rangle \\ &= \int_0^\infty dx\psi_{|n\rangle}(x)\psi^*_{|m\rangle}(x) - \int_{-\infty}^0 dx\psi_{|n\rangle}(x)\psi^*_{|m\rangle}(x) \\ &= \sqrt{\frac{2}{\pi}}\langle n|\hat{\sigma}_x|m\rangle. \qquad (3)\end{aligned}$$

Due to the non-unit factor $\sqrt{2/\pi}$ the detection is not perfect. The probability to produce the correct output given either eigenstate of $\hat{\sigma}_x$ is $\frac{1}{2}(1+\sqrt{2\eta_{hom}/\pi})$, where η_{hom} is the homodyne detection efficiency. For unit efficiency this number becomes $\sim 89.9\%$.

B. Pauli measurement via displacement and SPD

In Ref. [24] an alternative method was proposed for the implementation of measurements of $\hat{\sigma}_x$. In this setup the incoming field is displaced by mixing with a coherent state on a highly transmitting beam splitter and subsequently measured with a single-photon detector. For a real displacement α, the probability to observe no clicks at the detector (in the absence of loss), when an eigenstate of $\hat{\sigma}_x$ is incident, is given by

$$\begin{aligned}P(0|\pm) &= \left|\langle 0|D(\alpha)\left(|0\rangle \pm |1\rangle\right)/\sqrt{2}\right|^2 = \frac{1}{2}|\langle\alpha|0\rangle \pm \langle\alpha|1\rangle|^2 \\ &= \frac{1}{2}e^{-\alpha^2}\left(1\pm\alpha\right)^2, \qquad (4)\end{aligned}$$

where $D(\alpha)$ is the displacement operator and $|\alpha\rangle$ denotes a coherent state. Thus, by choosing $\alpha = -1$ one can ensure that the state $|0\rangle_x$ is faithfully detected, in the sense that it will always give a detector click, while the state $|1\rangle_x$ will give a no-click outcome with probability $2/e$. As we will see in the next section this has the same structure as an SPD $\hat{\sigma}_z$-measurement. However, this choice of α suggested in Ref. [24] is not necessarily optimal. When the single-photon detector has a non-unit efficiency η_{spd}, (4) becomes

$$P(0|\pm) = \frac{1}{2}e^{-\eta_{spd}\alpha^2}\left((1\pm\eta_{spd}\alpha)^2 + 1 - \eta_{spd}\right). \qquad (5)$$

The optimal choice of α will depend on η_{spd} as well as on the Bell scenario in which the approximate $\hat{\sigma}_x$-measurement is used.

III. GENERAL ERROR ANALYSIS

In any experiment, imperfections are likely to be present in the form of losses, detector inefficiencies, and noise. Depending on the particular setup, these imperfections might have different effects. We imagine Bell experiments in which each party chooses between measurements of the same two dichotomic observables, labeled by s and s'. For most cases we will consider, the settings s, s' will correspond to implementations of σ_z and σ_x respectively. A broad range of error behaviour can be modeled within the same framework, describing each measurement by a POVM with elements

$$\begin{aligned} M_\uparrow &= \eta_\uparrow |\uparrow\rangle_s\langle\uparrow| + (1-\eta_\downarrow)|\downarrow\rangle_s\langle\downarrow|, \\ M_\downarrow &= \eta_\downarrow |\downarrow\rangle_s\langle\downarrow| + (1-\eta_\uparrow)|\uparrow\rangle_s\langle\uparrow|, \end{aligned} \tag{6}$$

where $|\uparrow\rangle_s, |\downarrow\rangle_s$ are the eigenstates of the observable s. For ideal efficiencies $\eta_{\downarrow,\uparrow} = 1$ this POVM implements a projective measurement along the direction defined by s. For non-ideal efficiency we capture several types of errors as we now describe in detail.

One common loss-induced error occurs in experiments based on single-photon detection, where imperfect detectors lead to a decrease in the probability for observing one outcome (click) and an increased probability for the complementary outcome (no click). This makes sense e.g. for an optical implementation of (1), with the z-basis corresponding to the Fock basis and $|\uparrow\rangle_z = |1\rangle$. The vacuum never leads to clicks (in the absence of dark counts) while the single photon is detected with a finite efficiency. We can model this by taking

$$\eta_\downarrow = 1 \text{ and } \eta_\uparrow \in [0,1]. \tag{7}$$

Note that this models also describes the approximate implementation of σ_x via displacement, for $\alpha = \pm 1$, as discussed in Sec. II B.

Another relevant error consists in each input state being incorrectly identified as its opposite with some small probability. This corresponds for example to a polarisation measurement on a single photon, with a slight misalignment of the experimental measurement basis. As discussed in Sec. II A, it also occurs when homodyne measurements are used to approximate a measurement of σ_x. We can model this error by

$$\eta_\downarrow = \eta_\uparrow \in [0,1]. \tag{8}$$

Combining the POVMs (7) and (8), several interesting scenarios can be described. For example, taking (7) with $\eta_\uparrow = \eta$ for $\hat\sigma_z$-measurements and (8) with $\eta_\uparrow = (1+\sqrt{\eta})/2$ for $\hat\sigma_x$-measurements models an experiment where the state is subject to an amplitude damping (AD) channel which incoherently replaces $|1\rangle$ by $|0\rangle$ with probability $1-\eta$, and where all parties measure $\hat\sigma_z$, $\hat\sigma_x$. Such an AD channel could describe e.g. transmission loss affecting a single-photon W-state. Another example is the dephasing (D) channel, which is relevant for trapped-ion experiments, where dephasing is caused by magnetic-field and laser-intensity fluctuations, or spontaneous emission during Raman transitions [31], and also for photonic polarization-qubits [32]. For $\hat\sigma_z$- and $\hat\sigma_x$-measurements, the D channel with phase-flip probability $1-\eta$ is modeled by (8) with $\eta_\uparrow = 1$ and $\eta_\uparrow = (1+\eta)/2$ respectively.

One can imagine other scenarios for which measurements return a result that does not correspond to any of the binary outcomes, e.g. no click does not correspond to any polarization direction. To treat such an additional outcome one can either adopt a binning strategy, grouping it with one of the binary outcomes, or one can consider a setting with more outcomes, for which two-outcome Bell inequalities then no longer apply. Binning can be modelled within the POVM framework. For example (7) can be seen as binning of no-click events with the $\downarrow$-outcome. We will return to additional outcomes in V.

IV. BELL INEQUALITIES APPROACH

In this section we analyze the feasibility of the N modes W-state and possibly entangled with an additional atomic system to perform loophole free nonlocality tests using several Bell inequalities.

The atomic party is assumed to be able to perform ideal measurements in any basis. In order to find the best thresholds we optimise over these bases and also on the coefficient θ that describes the atom-photon entangled state (2). The photonic parties are assumed all to perform the same measurements: single-photon detection with efficiency described by eq. (7) or an approximate implementation of $\cos(\phi)\hat\sigma_x + \sin(\phi)\hat\sigma_y$. Actually, in all the cases considered, we found numerically that the best thresholds for the efficiencies are obtained for $\phi = 0$, that is for $\hat\sigma_x$ measurements.

A. Cabello et al. inequality

To begin with we investigate the inequality proposed for the 3-qubit W-state by Cabello in Ref. [25] and later generalised to more parties in Ref. [21]. Writing $p(o|s)$ for the probability that outcome $o \in \{0,1\}$ is obtained with the measurement setting $s \in \{\hat\sigma_x, \hat\sigma_z\}$, in our variant the inequality takes the form

$$p(\text{all } 0|\text{all } z) + p(\text{all but one } 0|\text{all } z) - p(x\text{'s different, all } z\text{'s } 0|\text{two } x, \text{ all else } z) - p(\text{all equal}|\text{all } x) \leq 0, \tag{9}$$

or more formally $B \leq 0$ with

$$B = p(0,\ldots,0|z,\ldots,z) + \sum_{\nu} p(\nu(1,0,\ldots,0)|z,\ldots,z) - \sum_{\nu'} p(\nu'(1,0,\ldots,0)|\nu'(x,x,z\ldots,z)) \\ - p(0,\ldots,0|x,\ldots,x) - p(1,\ldots,1|x,\ldots,x), \tag{10}$$

where ν runs over all cyclic permutations of length N and ν' runs over all permutations giving distinct images of $(1,2,0,\ldots,0)$. We note that the first term on the left-hand side was not present in previous works [21, 24, 25]. However, it is easy to check that the inequality is still valid, and since the term is always positive it can only increase any violation of the inequality. The violation of (10) attained by an N-party W-state approaches 1 exponentially fast with N, specifically

$$B(|W\rangle) = 1 - \frac{N}{2^{N-1}}. \tag{11}$$

In Ref. [21, 25] no analysis of the robustness of the violation was presented. From the growing violation and the robustness of the W-state entanglement against losses one might speculate that there could be experimental advantage to increasing the number of parties. As an illustration consider the N-party W-state undergoing amplitude damping in each of its qubits, thus becoming

$$\Phi_{amp}(|W\rangle\langle W|) = \eta|W\rangle\langle W| + (1-\eta)|vac\rangle\langle vac|, \tag{12}$$

where $|vac\rangle$ denotes the state with all modes in $|0\rangle$, and $1-\eta$ is the damping rate (or equivalently the loss probability). To assess the effect of the damping, we need to compute the value of B for the state $|vac\rangle$, and we find

$$B(|vac\rangle) = 1 - \frac{1}{2}\begin{pmatrix} N \\ 2 \end{pmatrix} - \frac{1}{2^{N-1}}. \tag{13}$$

From (11)-(13) the critical η below which no violation is possible can be analytically calculated and is given by

$$\tilde{\eta}_{amp} = \frac{8 - 2^{N+2} + 2^N N(N-1)}{(N-1)\,(2^N N - 8)}. \tag{14}$$

This is a monotonously decreasing function for $N > 2$ and hence the robustness to amplitude damping *decreases* with the number of parties. Considering general detection inefficiencies given by (7) and (8) for $\hat{\sigma}_z$ and $\hat{\sigma}_x$ respectively, one obtains Fig. 1, which also demonstrates that the robustness of the Cabello inequality decreases appreciatively for larger N. For the optimal case $N = 3$, to obtain a loophole-free test with homodyne realisations of $\hat{\sigma}_x$ and SPD realisations of $\hat{\sigma}_z$ we see that $\eta_{spd} > 86.3\%$ is required. For a displacement realisation of $\hat{\sigma}_x$ one can show that the bound is 86.4%.

B. Tight Bell inequalities

The inequality (10) is not a facet of the local polytope, that is, it is not a tight Bell inequality. So one should expect that the results obtained in the previous section are not optimal and that one could improve the thresholds testing the nonlocality of the W-states by means of tight Bell inequalities. To probe this we first investigate the tripartite scenario, which is fully characterized by the Sliwa inequalities [28]. For higher N one can consider the WWWZB-inequality [33] that fully describes tight Bell inequalities that can be formed with only full correlators, that is, only correlations between all subsystems. The WWWZB inequality describes the multipartite case where each party chooses between two different dichotomic measurements. There are 2^{2^N} tight, linear Bell inequalities which can be succinctly expressed in terms of the single non-linear inequality

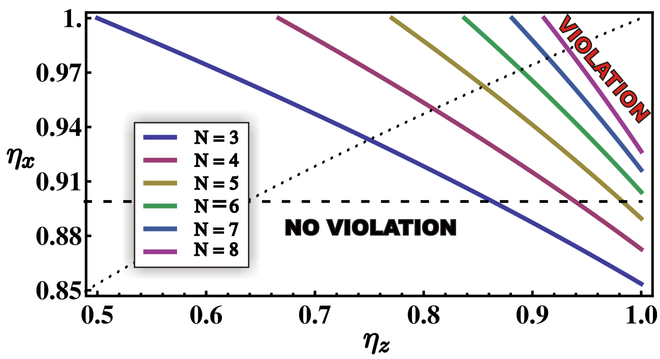

FIG. 1: Regions of violation of the Cabello inequality (10) for $N = 3,\ldots,8$ (bottom to top), when the error behaviour of $\hat{\sigma}_z$- and $\hat{\sigma}_x$-measurements is given respectively by (7) and (8). The inequality is violated above the solid lines. Intersections of the dotted and dashed with the solid lines give respectively the critical survival probability for the AD channel and the critical $\hat{\sigma}_z$-efficiency when $\hat{\sigma}_x$ is approximated via homodyning.

$$\sum_r \left|\widehat{\xi}(r)\right| \leq 1, \tag{15}$$

where $\widehat{\xi}(r) = 2^{-N}\sum_s (-1)^{r\cdot s}\,|\xi(s)|$, r and s are vectors describing a binary strings, for example $s = (s_1,\cdots,s_N)$ with $s_k = \{0,1\}$ and $\xi(s)$ is the full correlator so that s_k indicates the choice of the observable $A_k(s_k)$ at site k. In the following we analyze the pure photonic case and the atom-photon case.

Pure Photonic. We found that for $N = 3$ the Mermin inequality [29] - a particular case of the Sliwa and WWWZB inequalities - is the one providing the best robustness to losses and detection inneficiencies. However, for $N \geq 5$ the Mermin inequality is not violated any-

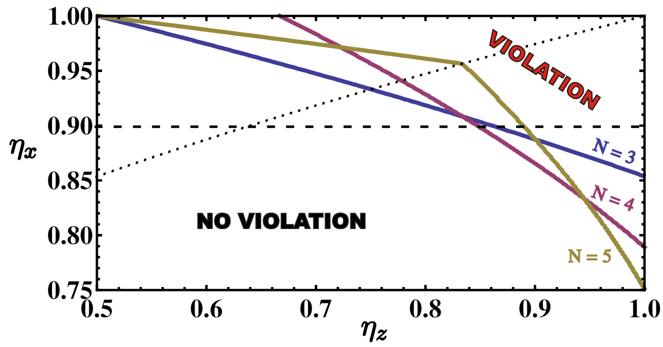

FIG. 2: Region of violations for the WWWZB inequality using an optical W-state for $N = 3, 4, 5$. The dashed and dotted lines indicate η_x as a function of η_z for the approximate σ_x implementation of Sec. II A and amplitude damping respectively. With homodyning, the best critical SPD efficiency is obtained for $N = 4$ or 7. When displacements are used, the best is $N = 4$.

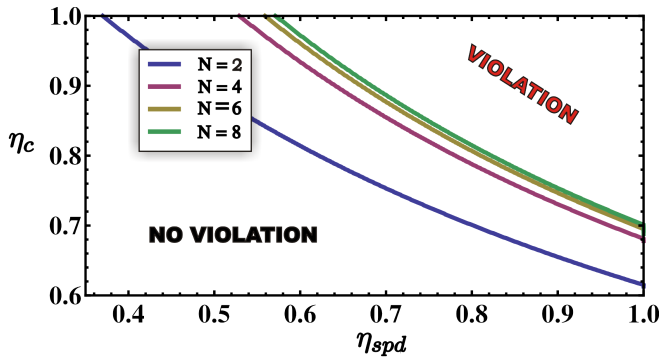

FIG. 3: Regions of violations of the WWWZB inequality for an optical W-state entangled with an atom for $N = 2, 4, 6, 8$ (bottom to top). The approximate σ_x implementation of Sec. II A with perfect homodyning is assumed. One sees that the optimal case is $N = 2$, which implies a very small value for the atom-photon entanglement.

more for the chosen measurement settings, even without any imperfections, and instead we consider the criteria (15). The results obtained are displayed in Fig. 2. It is interesting to note that with WWWZB inequalities, when $\eta_z \approx 1$ the required efficiencies for η_x significantly decrease with increasing N. For the particular case of SPD z-measurements and approximate x-measurements based on ideal homodyning, i.e. $\eta_x = \frac{1}{2}(1 + \sqrt{2/\pi})$, we find that the best threshold for the SPD efficiency is $\eta_{spd}^{\text{crit}} \approx 85\%$ obtained for $N = 4$ or 7. Similarly, when the x-measurement is based on displacement followed by SPD, the best threshold is again 85% found for $N = 4$. Thus, the performance of the two schemes is similar. However, we note that while $\eta_{hom} = 1$ was assumed for the homodyning scheme, there is no such assumption for the displacement scheme. Also, while the former scheme requires a fast switching between homodyning and single-photon detection, in the latter all that is needed is a fast transition between on and off states for the coherent displacement field, which may be easier experimentally (optimal values of α correspond to mean photon numbers on the order of 1). Hence, the displacement scheme may be the more attractive option for implementation.

Atom-Photon. Considering the case where the photonic modes are entangled with an atom the thresholds can be considerably improved. We compute the thresholds for violation of (15) in the particular case where x-measurements are implemented via ideal homodyning. The results are shown in Fig. 3 for $N = 2, 4, 6, 8$. One sees that the optimal thresholds are obtained for $N = 2$, a case already considered in Ref. [17]. For perfect coupling between the atom and the photon, a threshold as low as $\eta_{spd}^{\text{crit}} \approx 37\%$ can be obtained in the limit $\theta \to 0$, that is the optimal state is almost separable, in analogy with the Eberhard result [18]. For $N = 3$ on the other hand, the threshold is higher, $\eta_{spd}^{\text{crit}} \approx 55\%$, but the optimal state is more entangled. We find $\theta \approx -0.78$ which corresponds to a negativity of $\mathcal{N} = 0.993$ between the atom and the photonic modes ($\mathcal{N}$ ranges from 0 for a separable state to 1 for a maximally entangled state [34]). Similar results hold for higher N. We note that we are not optimizing over the coefficients that define the photonic state, e.g. the parameters of beam splitters utilized to divide the single photon between different modes. In principle if such an optimization is done, lower values for the minimum efficiencies can be achieved and in the limit where only one photonic mode is populated we effectively fall back into $N = 2$.

To compare the homodyning and displacement schemes for x-measurements, we have performed a more detailed calculation for the bipartite case, where the only relevant Bell inequality is CHSH. We first note, that when all detectors as well as the atom-photon coupling are taken to be perfect, the magnitude of the violation is higher in the displacement scheme. The schemes violate the LHV bound of 2 by 2.64 and 2.56. Next, we consider a realistic case in which the detection of the atomic state is imperfect at behaves according to (7), which corresponds e.g. to a flourescence measurement where the fluorescing state is detected with a finite efficiency. We fix the atomic detection efficiency to be 95%. Similarly, we take the homodyne detection efficiency to be $\eta_{hom} = 98\%$. The result is shown in Fig. 4. Under these assumptions, we find that for low SPD efficiency, the best tolerance to coupling loss is obtained using the homodyning scheme, as in Ref. [17]. On the other hand, for a coupling efficiency below $\sim 70\%$, which may well be the case in experiment, the tolerance to SPD inefficiency is better in the displacement scheme. We remark that the SPD efficiency required for violation for coupling efficiencies around $\sim 65\%$ is high, but that detectors reaching efficiencies in the 90%-range have been demonstrated [35].

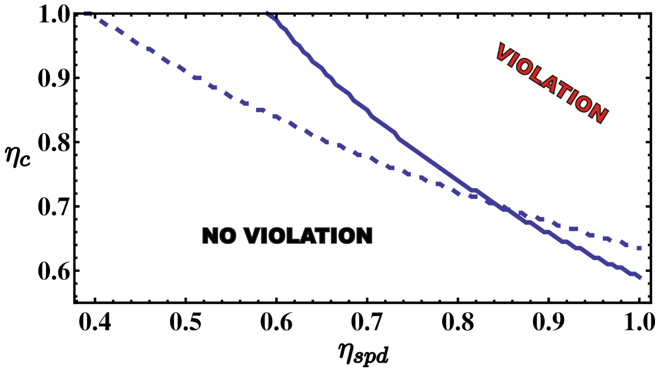

FIG. 4: Regions of violation of the CHSH inequality. The solid and dashed lines correspond to optical σ_x-implementations based on displacement and homodyning respectively. For the homodyne detectors $\eta_{hom} = 98\%$ while the atoms are detected with efficiency 95%.

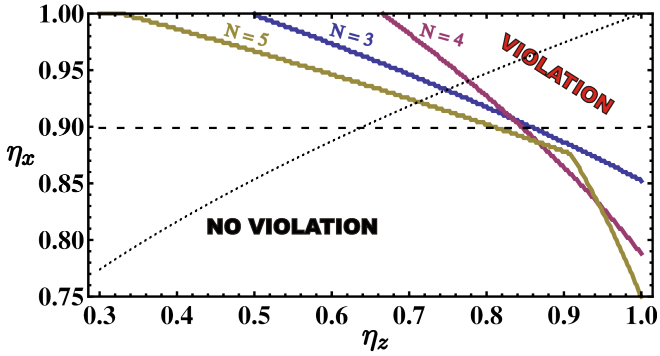

FIG. 5: Locality regions for the W-state under σ_z, σ_x measurements with error behaviour given by (7) and (8) respectively. Locality is violated above the solid lines. Intersections of the solid with the dashed and dotted lines respectively give the critical SPD efficiencies for the approximate σ_x implementation of Sec. II A and for amplitude damping.

V. POLYTOPE

To probe nonlocality in a more general scenario we consider a measure of nonlocality based on the Elitzur-Popescu-Rohrlich (EPR2) decomposition [27]. For a given experiment, that is for a fixed input state and set of measurement basis, we can compute the joint probability distribution of the outcomes given the settings $P = p(o_1, \ldots, o_N|s_1, \ldots, s_N)$. If this distribution is nonlocal, then the experiment must necessarily violate some Bell inequality. Conversely, if it is local no Bell inequality can be violated. Any such P can be decomposed into the convex mixture of a purely local and a purely nonlocal part

$$P = (1 - p_{NL})P_L + p_{NL}P_{NL}, \tag{16}$$

where P_L and P_{NL} are respectively a local and a non-signalling distribution. The minimal value $\tilde{p}_{NL}$ of p_{NL} over all such possible decompositions provides an unambiguous quantification of the nonlocality, called the *nonlocal content* of P.

The nonlocal content can be efficiently calculated by linear programming [7], and we make use of this to obtain critical efficiencies required for detection of nonlocality under various errors. However, let us first notice that the violation of any particular Bell inequality allows one to obtain a nontrivial lower bound on the nonlocal content. Indeed, for any (linear) Bell inequality $\mathcal{I} \leq \mathcal{I}^L$, the optimal decomposition of P yields $\mathcal{I}(P) \equiv (1 - \tilde{p}_{NL})\mathcal{I}(\tilde{P}_L) + \tilde{p}_{NL}\mathcal{I}(\tilde{P}_{NL})$. Since $\tilde{P}_L$ cannot violate the Bell inequality and $\mathcal{I}(\tilde{P}_{NL})$ is bounded by its maximal algebraic value $\mathcal{I}^{max}$, it follows that

$$\tilde{p}_{NL} \geq \frac{\mathcal{I}(P) - \mathcal{I}^L}{\mathcal{I}^{max} - \mathcal{I}^L}. \tag{17}$$

From here, it follows that any P which saturates the maximal algebraic value, $\mathcal{I}(P) = \mathcal{I}^{max}$ is automatically fully nonlocal, i.e. it has $\tilde{p}_{NL} = 1$. This is precisely what happens to GHZ states [30], which reach the algebraic maximum of the Mermin inequality [29] and are said to be fully nonlocal. We can now see that the W-state has a similar property for large N. On the one hand, the algebraic maximum of (10) is $\mathcal{I}^{max} = 1$, since the only positive terms in the inequality are mutually exclusive events that maximally sum up to one. On the other hand, from (11) the W-state violation approaches 1 exponentially fast, $\mathcal{I}(W) \to 1$, as $N \to \infty$. Thus, the nonlocal content of the W-state approaches unity.

Let us now turn to investigate the robustness of the nonlocality, by means of the polytope approach. We will look at the symmetric W-state (1), for the case of a single-photon implementation with imperfect photodetection as well as the case where losses are treated as a third outcome, corresponding e.g. to an implementation with polarisation qubits.

Single-photon case. We map out the regions of locality when all parties perform imperfect measurements of σ_z and σ_x, with the behavior of the imperfections given by (7) and (8) respectively. The result is shown in Fig. 5, for $N = 3, 4$, and 5, and should be compared with Fig. 2. We note that the regions in the two plots are identical for $N = 3, 4$, i.e. the WWWZB-criterion, based on full correlators only, yields the optimal solution in these cases. However, for $N = 5$, the region obtained from the full polytope, while qualitatively similar to that found taking only full correlators, yields lower critical efficiencies as might be expected in general. One special case of the (7)+(8) model is amplitude damping with ideal detectors, for which $\eta_x = (1+\sqrt{\eta_z})/2$ as indicated by the dashed line in Fig. 5. In the case $N = 3$, the threshold for violation agrees with (14) obtained from the inequality (10), but higher N allows for better tolerance than this inequality predicts. Another case is the approximate implementation of σ_x given in Sec. II A. In this setting $\eta_x \approx 89.9\%$, which we note falls in a region with only very modest improvement in critical efficiency up to $N = 5$. It is very

interesting to note that within the polytope approach, when $\eta_x \approx 1$ the required efficiencies for η_z significantly decrease with increasing N and for example for $N = 5$ we have $\eta_z^{\rm crit} \approx 33\%$.

Loss as 3rd outcome. In some cases, while inefficiencies may not be avoidable it can be possible to detect when a loss occurs. E.g. for measurements on optical polarisation qubits, a no-click event indicates that a photon was lost. For such a scenario, the no-click event can be treated as an additional outcome and by considering a polytope for distributions with three outcomes, the locality regions can be computed. We consider the ideal W-state (1) and two lossy Pauli measurements, the same for all parties. When the measurement are σ_x and σ_z, we find that locality is violated whenever $\eta_x > 2(1-\eta_z)$ for $N = 3, 4$. In particular, in the limit $\eta_z \to 1$, any non-zero η_x is sufficient for violation. This is analogous to the result obtained by Garbarino in the bipartite scenario [36]. The same holds when the two bases are arbitrary, but the same for all parties. Due to the limitations on our numerical algorithm, we were not able to go to $N > 4$.

VI. CONCLUSION

In this paper we have given a detailed analysis of the requirements to perform a nonlocality test using a single-photon W-state shared between multiple parties, and possibly entangled with an additional atomic system. We have focused on implementations using two measurement bases, of which one corresponds exactly to single-photon detection while the other requires and approximate implementation, e.g. via homodyning. Through a POVM model capturing a broad range of detection imperfections and loss we have numerically obtained the thresholds necessary for a loophole-free violation of local realism. In the case of atom-photon entanglement, comparing different numbers of parties N, we have shown that the bipartite scenario yields the best tolerance to loss. For high atom-photon coupling efficiency, the scheme considered very recently in Ref. [17], using both homodyne and single-photon detectors, yields the lowest thresholds for the single-photon detection efficiency, while for lower coupling efficiency another scheme based on displacements and single-photon detection performs better.

With offset in the Bell inequality of Ref. [25] we have demonstrated that for increasing N, the local content of the W-state under ideal Pauli measurements tends exponentially fast to zero, that is, the state becomes genuinely nonlocal. Surprisingly, we could also show that despite of that, the robustness of the violation with respect to loss actually decreases rapidly with N. Taking a step up in terms of generality, we then considered the WWWZB-inequalities [33] which provide a compact description of all tight Bell inequalities formed only from full correlators. From these we again found that increasing N does not necessarily lead to better loss thresholds. For example, assuming one approximate Pauli measurement based on ideal homodyning and another based on imperfect single-photon detection, the best threshold is reached already for $N = 4$, giving $\eta_{spd}^{\rm crit} \approx 85\%$. Finally, we approached the W-state via the polytope of local distributions, covering all possible Bell inequalities and thus taking another step up in terms of generality (as well as numerical complexity). With this approach, we found that the loss thresholds for a single photon are similar to those obtained from WWWZB, though slightly better for $N \leq 5$. For implementations that permit the detection of loss events, such as those based on polarisation qubits, much better thresholds can be obtained, with the threshold for one basis approaching 0 when the efficiency in the complementary basis is high.

We note that in the cases where increasing N leads to unchanged or worse loss thresholds, the scaling is not severe. This is a positive result which means that multipartite nonlocality tests become feasible with only slight improvements in detection efficiencies over those required for optimal N. As a possible extension of the results presented here, one can consider nonlocality tests involving more than a single photon and that can also be feasible prepared with current technology [15]. Also to consider more general scenarios where each party is allowed to perform more than just two measurements [37] can be a interesting approach to obtain less demanding detection efficiencies.

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